

Perturbation approaches and Taylor series

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Abstract

We comment on the new trend in mathematical physics that consists of obtaining Taylor series for fabricated linear and nonlinear unphysical models by means of homotopy perturbation method (HPM), homotopy analysis method (HAM) and Adomian decomposition method (ADM). As an illustrative example we choose a recent application of the HPM to a dynamic system of anisotropic elasticity.

1 Introduction

In the last years there has been great interest in the application of approximate variational and perturbation methods that lead to power-series solutions for linear and nonlinear problems in mathematical physics [1–13]. For

example, Chowdhury and Hashim [3] applied the powerful homotopy perturbation method (HPM) to obtain the Taylor series about $x = 0$ of the functions $y(x) = e^{x^2}$, $y(x) = 1 - x^3/3!$, $y(x) = \sin(x)/x$, and $y(x) = x^2 + x^8/72$. By means of the same method Chowdhury et al [2] derived the Taylor series about $t = 0$ for the solutions of the simplest population models. Bataineh et al [1] went a step further and resorted to the even more powerful homotopy analysis method (HAM) and calculated the Taylor expansions about $x = 0$ of the following two-variable functions: $y(x, t) = e^{x^2 + \sin t}$, $y(x, t) = x^2 + e^{x^2 + t}$, $y(x, t) = x^3 + e^{x^2 - t}$, $y(x, t) = t^2 + e^{x^3}$, and most impressive: $y(x, t) = -2 \ln(1 + tx^2)$ and $y(x, t) = e^{-tx^2}$. Zhang et al [12] dared to meddle with functions as complex as $u(x, t) = -2 \sec h^2[(x - 2t)/2]$, and $u(x, t) = -(15/8) \sec h^2[(x - 5t/2)]$. Because of the inherent difficulty in such functions the authors pushed the HPM calculation just to first order in t (I mean only the linear term of the Taylor series about $t = 0$). Bataineh et al [6] showed great insight and modified HAM to produce MHAM and found the Taylor expansions of the functions $u(t) = t^2 - t^3$ and $u(t) = 1 + t^2/16$. Sadighi and Ganji [11] calculated the Taylor expansions about $t = 0$ of $u(x, t) = 1 + \cosh(2x)e^{-4it}$ and $u(x, t) = e^{3i(x+3t)}$ by means of HPM and Adomian decomposition method (ADM), and verified that the results were exactly the same!! These authors dared to face the fact that $i^2 = -1$. By means of HPM Rafiq et al [10] also derived polynomial functions like $y(x) = x^4 - x^3$, $y(x) = x^2 + x^3$ and $y(x) = x^2 + x^8/72$. Özis and Agirseven [9] astonished the mathematics and physics community by expanding $u(x, t) = x^2 e^t$, $u(x, y, t) = y^2 \cosh t + x^2 \sinh t$ (and other such functions) about $t = 0$ by means of the HPM. Bataineh et al [5] used HAM to ob-

tain expansions about $x = 0$ for $w(x, t) = xe^{-t} + e^{-x}$, $w(x, t) = e^{x+t+t^2}$, $w(x, t) = e^{t+x^2}$ and $w(x, t) = e^{t^2+x^2}$. They thus managed to reproduce earlier ADM results. Although the authors did not state it explicitly in our opinion one of their achievements was to prove that the Taylor series have exactly the same form in different world locations.

There are many more articles where the authors apply HPM, HAM and ADM and produce results that an undergraduate student would easily obtain by means of a straightforward Taylor expansion of the model differential equations. It is a new trend in mathematical physics and we have just collected some examples published in only one journal.

In a recent paper Koçak and Yıldırım [13] applied HPM to a 3D Green's function for the dynamic system of anisotropic elasticity. The purpose of this article is to show the close connection between their results and the Taylor series approach. In Sec. 2 we introduce the problem and outline the application of the HPM. In Sec. 3 we apply the Taylor-series approach to the same problem. Finally in Sec. 4 we discuss the results and draw conclusions.

2 The homotopy perturbation method

Koçak and Yıldırım [13] considered equations of the form

$$\rho \frac{\partial^2 \mathbf{u}(\mathbf{x}, t)}{\partial t^2} = \hat{L}\mathbf{u}(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t) \quad (1)$$

where $\mathbf{u}, \mathbf{f} \in \mathbf{R}^m$ are vector functions of time and the spatial variables $\mathbf{x} \in R^n$, $\rho \in \mathbf{R}^{m \times m}$ is an invertible constant matrix and \hat{L} a differential operator independent of t .

They introduced a perturbation parameter p into the equation (1)

$$\rho \frac{\partial^2 \mathbf{u}(\mathbf{x}, t)}{\partial t^2} = p \left[\hat{L} \mathbf{u}(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t) \right] \quad (2)$$

and solved it by means of straightforward perturbation theory (with the fancy name of HPM)

$$\mathbf{u}(\mathbf{x}, t) = \sum_{j=0}^{\infty} p^j \mathbf{u}^{(j)}(\mathbf{x}, t) \quad (3)$$

Thus, they obtained

$$\begin{aligned} \frac{\partial^2 \mathbf{u}^{(0)}(\mathbf{x}, t)}{\partial t^2} &= \mathbf{0} \\ \frac{\partial^2 \mathbf{u}^{(j)}(\mathbf{x}, t)}{\partial t^2} &= \rho^{-1} \left[\hat{L} \mathbf{u}^{(j-1)}(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t) \delta_{j1} \right], \quad j = 1, 2, \dots \end{aligned} \quad (4)$$

Those authors were mainly interested in the Green's function for the differential equation (1) in which case $\mathbf{f}(x, t)$ is a product of delta functions.

3 Power series approach

Instead of applying the fashionable HPM we may try the well-known Taylor series approach and expand the functions in equation (1) in a Taylor series about $t = 0$:

$$\begin{aligned} \mathbf{u}(\mathbf{x}, t) &= \sum_{j=0}^{\infty} t^j \mathbf{u}_j(\mathbf{x}) \\ \mathbf{f}(\mathbf{x}, t) &= \sum_{j=0}^{\infty} t^j \mathbf{f}_j(\mathbf{x}) \end{aligned} \quad (5)$$

Thus we obtain a simple expression for the coefficients of the solution:

$$\mathbf{u}_{j+2} = \frac{1}{(j+1)(j+2)} \rho^{-1} \left(\hat{L} \mathbf{u}_j + \mathbf{f}_j \right), \quad j = 0, 1, \dots \quad (6)$$

so that

$$\begin{aligned} \mathbf{u}(\mathbf{x}, t) = & \mathbf{u}_0(x) + t\mathbf{u}_1(\mathbf{x}) + \frac{t^2}{2}\rho^{-1} \left(\hat{L}\mathbf{u}_0 + \mathbf{f}_0 \right) + \\ & \frac{t^3}{6}\rho^{-1} \left(\hat{L}\mathbf{u}_1 + \mathbf{f}_1 \right) + \dots \end{aligned} \quad (7)$$

One easily verifies that all the results shown by Koçak and Yıldırım [13] are particular cases of equation (7) with $\mathbf{u}_0 = \mathbf{f}_0 = 0$ and $\hat{L}\mathbf{u}_1 = -\mathbf{f}_1$ so that in every case they got the exact solution $\mathbf{u}(\mathbf{x}, t) = t\mathbf{u}_1(\mathbf{x})$. As an illustrative example consider their illustrative example [13]

$$\begin{aligned} \hat{L} &= \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \\ f(x, t) &= -2t \cos^2(x_1) \cos(x_2) + 3t \sin^2 \cos(x_2) \\ u_0(\mathbf{x}) &= 0, \quad u_1(\mathbf{x}) = \sin^2(x_1) \cos(x_2) \end{aligned} \quad (8)$$

with the exact solution $u(x, t) = t \sin^2(x_1) \cos(x_2)$ that clearly shows what we have stated above.

At this point we want to mention a salient characteristic of the new trend mentioned above: the authors fabricate unphysical problems with exact solutions and solve them by means of methods like ADM, HPM, and HAM.

4 Further comments and Conclusions

The discussion above clearly shows that the paper of Koçak and Yıldırım [13] is another example of the new trend in mathematical physics that consists of obtaining the textbook Taylor series by means of elaborate approximation methods. We proved that this standard approach enables one to solve their differential equations exactly. However, those authors did not obtain the

exact result even at fifth order of HPM. The reason appears to be a wrong calculation of the correction of first order. For example, in the introductory simple equation they obtained $u^{(0)} = 0$ and $u^{(1)} = \Theta(t)t\delta(x)$, where $\Theta(t)$ is the Heaviside step function, but this solution does not satisfy the differential equation $\partial^2 u^{(1)}/\partial t^2 = \delta(t)\delta(x)$. It is wrong by a factor two and seems to be the reason why Koçak and Yıldırım [13] did not obtain the exact result at second order as predicted by the straightforward Taylor series. However, this sloppiness is weightless when contrasted with such magnificent contribution to nowadays science and the referees and editors of the journal are content with it.

Another issue is the physical utility of the models considered by Koçak and Yıldırım [13]. Notice, for example, that the amplitude of the solution $\mathbf{u}(\mathbf{x}, t)$ given by equations (20)–(23) in that paper, which is of the form $\mathbf{u}(\mathbf{x}, t) = t\mathbf{u}_1(\mathbf{x})$ discussed above in Sec. 3, increases unboundedly with time. But the new trend will not be deterred by such prosaic considerations.

The reader may find the discussion of other articles that belong to the new trend in mathematical physics elsewhere [14–22]. We recommend the most interesting case of the predator–prey model that predicts a negative number of rabbits [17].

Finally, we mention that a slightly different version of this comment was rejected on the basis that “I have determined that it lacks the qualities of significant timeliness and novelty that we are seeking in this journal. We request you to consider submitting your manuscript in another forum that would better suit the material your manuscript covers”. After pondering a while we realized that we submitted our manuscript a couple of weeks after

the appearance of that paper. The new trend is advancing so fast that we delayed rather too much.

References

- [1] A. Sami Bataineh, M. S. M. Noorani, and I. Hashim, Phys. Lett. A 371 (2007) 72-82.
- [2] M. S. H. Chowdhury, I. Hashim, and O. Abdulaziz, Phys. Lett. A 368 (2007) 251-258.
- [3] M. S. H. Chowdhury and I. Hashim, Phys. Lett. A 365 (2007) 439-447.
- [4] M. Esmailpour and D. D. Ganji, Phys. Lett. A 372 (2007) 33-38.
- [5] A. Sami Bataineh, M.S.M. Noorani, and I. Hashim, Phys. Lett. A 372 (2008) 613-618.
- [6] A. Sami Bataineh, M.S.M. Noorani, and I. Hashim, Phys. Lett. A 372 (2008) 4062-4066.
- [7] M. S. H. Chowdhury and I. Hashim, Phys. Lett. A 372 (2008) 1240-1243.
- [8] M. Inc, Phys. Lett. A 372 (2008) 356-360.
- [9] T. Özis and D. Agirseven, Phys. Lett. A 372 (2008) 5944-5950.
- [10] A. Rafiq, M. Ahmed, and S. Hussain, Phys. Lett. A 372 (2008) 4973-4976.
- [11] A. Sadighi and D. D. Ganji, Phys. Lett. A 372 (2008) 465-469.

- [12] B.-G. Zhang, S.-Y. Li, and Z.-R. Liu, Phys. Lett. A 372 (2008) 1867-1872.
- [13] H. Koçak and A. Yıldırım, Phys. Lett. A 373 (2009) 3145-3150.
- [14] F. M. Fernández, Perturbation Theory for Population Dynamics, arXiv:0712.3376v1
- [15] F. M. Fernández, On Some Perturbation Approaches to Population Dynamics, arXiv:0806.0263v1
- [16] F. M. Fernández, On the application of homotopy-perturbation and Adomian decomposition methods to the linear and nonlinear Schrödinger equations, arXiv:0808.1515v1
- [17] F. M. Fernández, On the application of the variational iteration method to a prey and predator model with variable coefficients, arXiv:0808.1875v2
- [18] F. M. Fernández, On the application of homotopy perturbation method to differential equations, arXiv:0808.2078v2
- [19] F. M. Fernández, Homotopy perturbation method: when infinity equals five, 0810.3318v1
- [20] F. M. Fernández, Phys. Scr. 79 (2009) 055003 (2pp.).
- [21] F. M. Fernández, Amazing variational approach to chemical reactions, arXiv:0906.0950v1 [physics.chem-ph]

- [22] F. M. Fernández, On the homotopy perturbation method for Boussinesq-like equations, arXiv:0907.4481v1 [math-ph]